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On a Stationary Solution for the Magnetohydrodynamic Equations in a Bounded Domain

Norikazu YAMAGUCHI

E-mail: norikazu@edu.u-toyama.ac.jp

Dedicated to the memory of Professor Tetsuro Miyakawa

Abstract. A stationary problem of the magnetohydrodynamic (MHD) equations in three dimensional bounded domain is considered. The MHD system is known as a mathematical model for the motion of viscous, incompressible and electrically conducting fluid and as a hydrodynamic model for the motion of plasma. We obtained a result concerning existence and uniqueness for the stationary problem provided that viscosity and conductivity of fluid satisfy suitable smallness conditions.
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§ 1. Introduction and main result

1.1. Physical background and problem

The main objective of the present article is the motion of viscous, incompressible and electrically conducting fluid, e.g., mercury.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain whose boundary $\partial\Omega$ is of class C^2. The stationary motion of the above fluid in Ω is governed by the magnetohydrodynamic (MHD) equations concerning the velocity $\boldsymbol{u} = (u^1(x), u^2(x), u^3(x))$, pressure $\pi = \pi(x)$ and magnetic flux density $\boldsymbol{b} = (b^1(x), b^2(x), b^3(x))$:

$$\begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla\pi + \boldsymbol{f} + \dfrac{1}{\mu}\boldsymbol{b}\wedge \operatorname{rot}\boldsymbol{b}, \\ -\dfrac{1}{\sigma\mu}\Delta\boldsymbol{b} - \operatorname{rot}(\boldsymbol{u}\wedge\boldsymbol{b}) = \boldsymbol{0}, \\ \operatorname{div}\boldsymbol{u} = 0, \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $\boldsymbol{f} = (f^1(x), f^2(x), f^3(x))$ is a given external force; ν, σ and μ are positive constants and stand for the viscosity, conductivity and permeability of the fluid, respectively.

(1.1) is derived from the Navier-Stokes equations with the Lorentz force, Maxwell's equations and Ohm's law under the *MHD approximation*. The MHD approximation is well acceptable, because the velocity of fluid is much slower than that of light. (1.1) is also known as a hydrodynamic model for the motion of plasma without the Hall effect and energy transfer between ions and electrons on collision. For details of physical background and derivation of (1.1), see Landau & Lifshitz [7, Chapter 8].

In order to determine the motion of fluid in Ω, we need some boundary conditions for $\boldsymbol{u}$ and $\boldsymbol{b}$ in addition to (1.1). On the boundary of Ω, we impose the following boundary conditions on $\boldsymbol{u}$ and $\boldsymbol{b}$.

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \\ \boldsymbol{n} \cdot \boldsymbol{b} = 0, \quad \operatorname{rot} \boldsymbol{b} \wedge \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here and hereafter $\boldsymbol{n}$ denotes the unit outer normal on $\partial\Omega$. Usually, $(1.2)_1$ is called the *non-slip* boundary condition and $(1.2)_2$ is called the *perfectly conducting wall*.

Instationary problem of MHD equations in the framework of L_r space is well studied by many researchers. For example, by Yoshida & Giga [13], Akiyama [2], Schonbek, Schonbek and Süli [8] and the author [11], well-posedness and asymptotic behavior of solution are obtained (see also the references therein). In other words, these results are investigations for asymptotic stability for the trivial stationary solution. However, as far as the author knows, there are a few results concerning the stability of *non-trivial* stationary flow in the framework of L_r space. The stability problem of non-trivial stationary flow is quite important not only in mathematics, but also in fluid mechanics and engineering. Thus, we shall investigate the stationary problem (SP): (1.1)-(1.2) as a starting point of mathematical analysis for the stability problem for the MHD flow in bounded container. Our main purpose of the present paper is to show that (SP) has a unique solution in L_3 space.

1.2. Notation

Before stating our result, here we shall introduce notation. In order to denote the vector field in $\mathbb{R}^3$, we use bold face like $\boldsymbol{u}$. For two vector fields $\boldsymbol{u}$ and $\boldsymbol{w}$, $\boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \wedge \boldsymbol{w}$ denote the usual inner- and exterior-product, respectively.

For differentiations of vector field $\boldsymbol{u}$ and scalar function π, we use the following symbols:

$$\partial_j \pi = \frac{\partial \pi}{\partial x_j}, \qquad \nabla \pi = (\partial_1 \pi, \partial_2 \pi, \partial_3 \pi),$$

$$\operatorname{div} \boldsymbol{u} = \sum_{j=1}^{3} \partial_j u^j, \quad \operatorname{rot} \boldsymbol{u} = (\partial_2 u^3 - \partial_3 u^2, \partial_3 u^1 - \partial_1 u^3, \partial_1 u^2 - \partial_2 u^1),$$

$$\Delta \boldsymbol{u} = \sum_{j=1}^{3} \partial_j^2 \boldsymbol{u} \quad \text{(the Laplace operator)}, \quad (\boldsymbol{w} \cdot \nabla)\boldsymbol{u} = \left(\sum_{j=1}^{3} w^j \partial_j \right) \boldsymbol{u}.$$

Let $L_r(\Omega)$ denote the usual Lebesgue space ($1 \le r \le \infty$) with norm $\| \cdot \|_r$, $W_r^m(\Omega)$ denote the L_r-Sobolev space of order m ($m \in \mathbb{N}_0$) and $W_{r,0}^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in $W_r^1(\Omega)$. $C_0^\infty(\Omega)$ is the set of all infinitely differentiable function in Ω with compact support (For details, see Adams & Fournier [1]). For function spaces of vector fields, we use the following symbols: $L_r(\Omega)^3 = \{\boldsymbol{u} \mid u^j \in L_r(\Omega), j = 1, 2, 3\}$, likewise $W_r^m(\Omega)^3$ and $C_0^\infty(\Omega)^3$.

In order to denote various constants, we use the same letters C and $C_{a,b,\ldots}$ which means that the constant depends on $a, b, \ldots$. The constants C and $C_{a,b,\ldots}$ may change from line to line .

1.3. Helmholtz decomposition and Stokes operators

To give an abstract form of (SP), here we shall introduce the Helmholtz decomposition of L_r-vector field. Let $1 < r < \infty$. As shown in Fujiwara and Morimoto [5], $L_r(\Omega)^3$ admits the Helmholtz decomposition.

$$L_r(\Omega)^3 = X_r(\Omega) \oplus \{\nabla\pi \mid \pi \in W_r^1(\Omega)\}, \quad \oplus : \text{direct sum},$$

where

$$\begin{aligned} X_r(\Omega) &= \overline{\{\boldsymbol{u} \in C_0^\infty(\Omega)^3 \mid \operatorname{div}\boldsymbol{u} = 0\}}^{\|\cdot\|_{L_r(\Omega)}} \\ &= \{\boldsymbol{u} \in L_r(\Omega)^3 \mid \operatorname{div}\boldsymbol{u} = 0, \boldsymbol{n}\cdot\boldsymbol{u}|_{\partial\Omega} = 0\}. \end{aligned} \tag{1.3}$$

Let $P = P_r$ be a continuous projection from $L_r(\Omega)^3$ into $X_r(\Omega)$ associated with the above decomposition. Then we shall define the Stokes operator with non-slip boundary condition $A = A_r$ associated with (1.1) and $(1.2)_1$ as follows.

$$\begin{gathered} A\boldsymbol{u} = -P\Delta\boldsymbol{u} \quad \text{for } \boldsymbol{u} \in D(A), \\ D(A_r) = X_r(\Omega) \cap W_r^2(\Omega)^3 \cap W_{r,0}^1(\Omega)^3. \end{gathered}$$

By a similar manner, we shall also define the Stokes operator with perfectly conducting wall $B = B_r$ associated with (1.1) and $(1.2)_2$ as follows.

$$\begin{gathered} B\boldsymbol{u} = \operatorname{rot}\operatorname{rot}\boldsymbol{u} \quad \text{for } \boldsymbol{u} \in D(B), \\ D(B_r) = X_r(\Omega) \cap \{\boldsymbol{u} \in W_r^2(\Omega)^3 \mid \operatorname{rot}\boldsymbol{u}\wedge\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}\}. \end{gathered}$$

Nothing the formula: $\Delta\boldsymbol{u} = \nabla\operatorname{div}\boldsymbol{u} - \operatorname{rot}\operatorname{rot}\boldsymbol{u}$, $\Delta\boldsymbol{u} = -\operatorname{rot}\operatorname{rot}\boldsymbol{u}$ holds for $\boldsymbol{u}$ satisfying $\operatorname{div}\boldsymbol{u} = 0$. It should be remarked that $D(B_r)$ includes all boundary conditions for the magnetic flux density: $(1.2)_2$, because of (1.3).

1.4. Main result

Using A and B, (SP) is rewritten in the following abstract form in $X_r(\Omega) \times X_r(\Omega)$.

$$\begin{cases} \nu A\boldsymbol{u} + P\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \dfrac{1}{\mu}\boldsymbol{b}\cdot\nabla\boldsymbol{b}\right) = P\boldsymbol{f}, \\ \dfrac{1}{\sigma\mu}B\boldsymbol{b} + \boldsymbol{u}\cdot\nabla\boldsymbol{b} - \boldsymbol{b}\cdot\nabla\boldsymbol{u} = \boldsymbol{0}. \end{cases} \tag{1.4}$$

Here we have used the elementary formulas of the vector calculus.

We are now ready to define the stationary solution to (SP) which we shall seek.

Definition 1.1 (stationary solution)**.** We call a pair of vector fields $(\boldsymbol{u}, \boldsymbol{b})$ stationary solution to (SP) of class R_0 if

$$\boldsymbol{u} \in D(A_3), \quad \boldsymbol{b} \in D(B_3), \quad \|\boldsymbol{u}\|_{D(A_3)} + \|\boldsymbol{b}\|_{D(B_3)} \le R_0,$$

and $(\boldsymbol{u}, \boldsymbol{b})$ enjoys

$$\begin{cases} \nu A\boldsymbol{u} + P\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \dfrac{1}{\mu}\boldsymbol{b}\cdot\nabla\boldsymbol{b}\right) = P\boldsymbol{f} \quad \text{in } X_3(\Omega), \\ \dfrac{1}{\sigma\mu}B\boldsymbol{b} + \boldsymbol{u}\cdot\nabla\boldsymbol{b} - \boldsymbol{b}\cdot\nabla\boldsymbol{u} = \boldsymbol{0} \quad \text{in } X_3(\Omega). \end{cases} \tag{1.5}$$

Remark 1.2. The reason why we seek the stationary solution in L_3 framework is that L_3 plays an important role in the study of asymptotic stability. To argue asymptotic stability of the stationary flow, initial data $(\boldsymbol{u}, \boldsymbol{b})|_{t=0}$ will be taken form the space $X_3(\Omega) \times X_3(\Omega)$. Therefore, it is convenient to construct a stationary solution in the same space as initial data.

We are now in a position to state our main result.

Theorem 1.3. *Let $\boldsymbol{f} \in L_3(\Omega)^3$. There exist a $\delta > 0$ such that if $\nu^{-1} < \min(\delta, 1)$ and $\sigma < \delta$, then* (SP) *has a unique stationary solution* $(\boldsymbol{u}, \boldsymbol{b})$ *of class R_0 with $R_0 = 2\|P\|_{\mathcal{L}(L_3(\Omega), X_3(\Omega))}\|\boldsymbol{f}\|_3$.*

§ 2. Proof of main result

This section is devoted to the proof of our main theorem.

It is well known that the operator A is invertible when Ω is bounded (see Giga [6] and Farwig & Sohr [4]). According to Akiyama, Kasai, Shibata & Tsutsumi [3]), the operator B is also invertible provided that Ω is *simply connected* and bounded. From a view point of the result due to Akiyama.*et.al.*, from now on we assume that Ω is bounded and simply connected.

Using A^{-1} and B^{-1}, we have the following abstract equations for $\boldsymbol{u}$ and $\boldsymbol{b}$ which is equivalent to (1.4).

$$\begin{cases} \boldsymbol{u} = \dfrac{1}{\nu} A^{-1} P \left(-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \dfrac{1}{\mu} \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \boldsymbol{f} \right), \\ \boldsymbol{b} = \sigma\mu B^{-1} \left(-\boldsymbol{u} \cdot \nabla \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{u} \right). \end{cases} \tag{2.1}$$

To show unique existence of (SP), it is sufficient to show that (2.1) has a unique solution.

Let us define the mapping $\Phi : D(A_3) \times D(B_3) \to D(A_3) \times D(B_3)$ by

$$\Phi \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\nu} A^{-1} P \left(-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \dfrac{1}{\mu} \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \boldsymbol{f} \right) \\ \sigma\mu B^{-1} \left(-\boldsymbol{u} \cdot \nabla \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{u} \right) \end{bmatrix}.$$

Then (1.5) is equivalent to the following nonlinear equation concerning $\boldsymbol{u}$ and $\boldsymbol{b}$.

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{b} \end{bmatrix} = \Phi \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{b} \end{bmatrix} \quad \text{in } D(A_3) \times D(B_3).$$

Our task is to find a fixed point of Φ. Once we get fixed point of Φ, such a fixed point gives a stationary solution of (SP). From a view point of Banach's fixed point theorem, it is enough to show that Φ is contractive on some complete metric space.

Theorem 1.3 is a direct consequence of the following proposition.

Proposition 2.1. *For $\boldsymbol{f} \in L_3(\Omega)^3$, there exists a $\delta > 0$ such that if ν and σ satisfy*

$$\frac{1}{\nu} < \max(\delta, 1), \qquad \sigma < \delta \tag{2.2}$$

then Φ is a contraction mapping on the complete metric space:

$$\mathcal{I}_{R_0} = \{(\boldsymbol{u}, \boldsymbol{b}) \in D(A_3) \times D(B_3) \mid \|\boldsymbol{u}\|_{D(A_3)} + \|\boldsymbol{b}\|_{D(B_3)} \le R_0\}.$$

Here R_0 is a constant satisfying $R_0 < 2C_0K$, where $C_0 = \|P\|_{\mathcal{L}(L_3(\Omega)^3, X_3(\Omega))}$ and $K = \|\boldsymbol{f}\|_3$.

Proof of Proposition 2.1. By the Hölder inequality and the Sobolev embedding relations, we have

$$\left\| P \left(-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\mu} \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \boldsymbol{f} \right) \right\|_3 \le C_0 C_1 \left(\|A\boldsymbol{u}\|_3^2 + \frac{1}{\mu} \|B\boldsymbol{b}\|_3^2 \right) + C_0 \|\boldsymbol{f}\|_3.$$

By a similar manner, we have

$$\|(-\boldsymbol{u}\cdot\nabla\boldsymbol{b}+\boldsymbol{b}\cdot\nabla\boldsymbol{u})\|_3 \le 2\mu C_1\left(\|A\boldsymbol{u}\|_3\|B\boldsymbol{b}\|_3\right).$$

Hence we obtain the following estimate.

$$\begin{aligned}\left\|\Phi\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{bmatrix}\right\|_{D(A_3)\times D(B_3)} &\le \frac{1}{\nu}\left(C_0C_1\|A\boldsymbol{u}\|_3^2+\frac{1}{\mu}\|B\boldsymbol{b}\|_3^2\right)+\frac{C_0}{\nu}\|\boldsymbol{f}\|_3+2\sigma\mu C_1\|A\boldsymbol{u}\|_3\|B\boldsymbol{b}\|_3\\ &\le C_2M_{\nu,\sigma}\left\|\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{bmatrix}\right\|^2_{D(A_3)\times D(B_3)}+C_0\frac{K}{\nu}.\end{aligned}$$

Here we have set

$$C_2=\max\left(C_0C_1, C_0C_1\frac{1}{\mu}, C_1\mu\right),\quad M_{\nu,\sigma}=\max\left(\frac{1}{\nu},\sigma\right).$$

Set

$$\delta=\frac{1}{4C_0C_2K},\quad R_0=\frac{1-\sqrt{1-4C_0C_2KM_{\nu,\sigma}\nu^{-1}}}{2C_2M_{\nu,\sigma}}.$$

Choose ν and σ in such a way that

$$\nu>1,\quad M_{\nu,\sigma}<\delta, \tag{2.3}$$

we see that $1-4C_0C_2KM_{\nu,\sigma}\nu^{-1}>0$ and

$$R_0=\frac{2C_0K}{1+\sqrt{1-4C_0C_2KM_{\nu,\sigma}\nu^{-1}}}<2C_0K. \tag{2.4}$$

Hence, by (2.3) and (2.4), we have

$$\left\|\Phi\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{bmatrix}\right\|_{D(A_3)\times D(B_3)} \le C_0\frac{K}{\nu}+C_2M_{\nu,\sigma}R_0^2<2C_0K$$

for any $(\boldsymbol{u},\boldsymbol{b})\in \mathcal{I}_{R_0}$. This implies that

$$\Phi\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{bmatrix}\in\mathcal{I}_{R_0}\quad\text{for any } \begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{bmatrix}\in\mathcal{I}_{R_0}.$$

In a similar manner, one can get

$$\begin{aligned}\left\|\Phi\begin{bmatrix}\boldsymbol{u}_1\\ \boldsymbol{b}_1\end{bmatrix}-\Phi\begin{bmatrix}\boldsymbol{u}_2\\ \boldsymbol{b}_2\end{bmatrix}\right\|_{D(A_3)\times D(B_3)} &\le 2C_2M_{\nu,\sigma}R_0(\|A(\boldsymbol{u}_1-\boldsymbol{u}_2)\|_3+\|B(\boldsymbol{b}_1-\boldsymbol{b}_2)\|_3)\\ &\le 2C_2M_{\nu,\sigma}R_0\left\|\begin{bmatrix}\boldsymbol{u}_1\\ \boldsymbol{b}_1\end{bmatrix}-\begin{bmatrix}\boldsymbol{u}_2\\ \boldsymbol{b}_2\end{bmatrix}\right\|_{D(A_3)\times D(B_3)}\end{aligned} \tag{2.5}$$

for $(\boldsymbol{u}_1,\boldsymbol{b}_1),(\boldsymbol{u}_2,\boldsymbol{b}_2)\in\mathcal{I}_{R_0}$. From (2.3) and (2.4), we see that

$$2C_2M_{\nu,\sigma}R_0<2C_2\frac{1}{4C_0C_2K}2C_0K<1.$$

Combining this fact and (2.5), we conclude that Φ is contraction mapping from $\mathcal{I}_{R_0}$ to $\mathcal{I}_{R_0}$. This completes the proof of Proposition 2.1. (Q.E.D.)

Since $\mathcal{I}_{R_0}$ is complete metric space, by virtue of Banach's fixed point theorem, Proposition 2.1 yields Theorem 1.3.

Remark 2.2. At the end of the present paper, let us give a brief remark on stationary solution which we have constructed. In construction of the stationary solution, we have essentially used the smallness condition for ν^{-1} and σ. Instead of such a smallness condition, we can show that a similar result of Theorem 1.3 by use of another smallness condition for the external force $\boldsymbol{f}$. More precisely, if we choose $\|\boldsymbol{f}\|_3$ sufficiently small, then Φ is contractive on $\mathcal{I}_{R_0}$ with $R_0 < 2C_0/\nu$.

However, in each case, obtained stationary solutions are small in some senses.

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